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Extremal Polynomials for Weighted Markov Inequalities

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1. INTRODUCTION

Let $\mathscr{P}_n$ denote the collection of real algebraic polynomials of degree $\leqslant n$, and $\mathscr{P}_n^*$ that subcollection of $\mathscr{P}_n$ consisting of the monic polynomials of degree $\leqslant n$. Let $\mathscr{W}$ denote the collection of real weight functions, w, such that: $w(x) > 0$ for all $x \in \mathscr{R}$, w' is continuous on $\mathscr{R}$, and $\lim[x^n w(x)] = \lim[x^n w'(x)] = 0$ as $|x| \to \infty$, $n = 1, 2, \ldots$. All norms considered in this paper are sup norms on $\mathscr{R}$ (i.e., $\|f\| = \sup\{|f(x)| : x \in \mathscr{R}\}$). For each $n = 1, 2, \ldots$, define

$$\lambda_n = \sup_{p \in \mathscr{P}_n^*} \frac{\|wp'\|}{\|wp\|} \qquad \text{and} \qquad \mu_n = \sup_{p \in \mathscr{P}_n^*} \frac{\|(wp)'\|}{\|wp\|}.$$

By standard arguments it can be shown that λ_n and μ_n are finite and that there exist polynomials $p, q \in \mathscr{P}_n^*$ for which $\|wp'\|/\|wp\| = \lambda_n$ and $\|(wq)'\|/\|wq\| = \mu_n$. We will refer to such polynomials p or q as extremal polynomials for λ_n or μ_n, respectively. Clearly the following inequalities of Markov type hold for all $p \in \mathscr{P}_n$:

$$\|wp'\| \leqslant \lambda_n \|wp\| \qquad \text{and} \qquad \|(wp)'\| \leqslant \mu_n \|wp\|.$$

Moreover λ_n and μ_n are the best possible constants in these inequalities. Estimates of λ_n and μ_n have been determined for various special weight functions (cf. [3, 6, 7]).

We also introduce the monic polynomials, T_n, of exact degree n, which are extremal in the sense that $\|wT_n\| = \inf\{\|w(x)[x^n - q(x)]\| : q \in \mathscr{P}_{n-1}\}$. Since $\{x^k w(x) : k = 0, 1, \ldots, n-1\}$ is a Haar system on $\mathscr{R}$, it is well known (cf. [1]) that T_n is uniquely characterized by the fact that wT_n has an alternant of size $n+1$. (An alternant of size N for a function, f, is a set

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of N points, $x_1 < \cdots < x_N$, such that $|f(x_k)| = \|f\|$, $k = 1, ..., N$ and $f(x_{k+1}) = -f(x_k)$, $k = 1, ..., N-1$. A maximal alternant for f is an alternant for f whose size is as large as possible.) It is known [4] that T_n is also extremal in the sense that among all the functions, wp, $p \in \mathscr{P}_n^*$, the one with the largest (or smallest) e-point is wT_n. (An e-point of a function, f, is a point, x_0, such that $|f(x_0)| = \|f\|$.) In other words, if a_n and b_n denote the smallest an largest e-points of wT_n then for any $p \in \mathscr{P}_n$, $\|wp\| = \max\{|w(x)\, p(x)|: a_n \leqslant x \leqslant b_n\}$. It is clear that λ_n, μ_n, T_n, a_n, and b_n depend on the weight function, w, but for simplicity our notations will not indicate this dependency.

The purpose of this paper is to prove

THEOREM 1. *Let $w \in \mathscr{W}$ and suppose $p \in \mathscr{P}_n^*$, $n \geqslant 2$, is any extremal for μ_n. Then*

(i) *A maximal alternant for wp is of size n or $n+1$.*

(ii) *If w'/w is decreasing on $\mathscr{R}$ then there is exactly one maximal alternant for wp. Moreover if this maximal alternant, $x_1 < \cdots < x_n$, is of size n (i.e., if $p \neq T_n$) then $(w\omega)'(t_0) = 0$, where $\omega(x) = (x - x_1) \cdots (x - x_n)$ and t_0 is any e-point of $(wp)'$.*

THEOREM 2. *Let $w \in \mathscr{W}$ and suppose $p \in \mathscr{P}_n^*$ is any extremal for λ_n.*

(i) *If $n = 1$ then $p = T_1$.*

(ii) *If $n \geqslant 2$ then a maximal alternant for wp is of size n or $n+1$.*

(iii) *If w'/w is decreasing on $\mathscr{R}$ then there is exactly one maximal alternant for wp. Moreover if this maximal alternant, $x_1 < \cdots < x_n$, is of size n (i.e., if $p \neq T_n$) then $\omega'(t_0) = 0$, where $\omega(x) = (x - x_1) \cdots (x - x_n)$ and t_0 is any e-point of wp'.*

THEOREM 3. *If $w(x) = \exp(-x^2)$ and $p \in \mathscr{P}_n^*$ is any extremal for μ_n, $n \geqslant 1$, then $p = T_{n-1}$ or $p = T_n$, where $T_0 := 1$.*

These theorems will be proved in Section 3, but first we need some preliminary results.

2. LEMMAS

LEMMA 1. *Suppose:*

(i) *$f \neq 0$ and g are real functions continuous on $[a, b]$,*

(ii) *$M = \max_{a \leqslant x \leqslant b} |f(x)|$ and $\mathscr{E} = \{x \in [a, b]: |f(x)| = M\}$,*

(iii) *there exists a set* $\mathscr{A} \supset \mathscr{E}$ *such that* $\mathscr{A}$ *is open relative to* $[a, b]$ *and* $f(x)\, g(x) \geqslant 0$ *for each* $x \in \mathscr{A}$.

Then for sufficiently small positive ε, $\max_{a \leqslant x \leqslant b} |f(x) - \varepsilon g(x)| \leqslant M$.

It should be noted that Lemma 1 is a slight variation of a more standard result which states that the inequality in the conclusion is strict if, instead of (iii), it is assumed that $f(x)\, g(x) > 0$ for each $x \in \mathscr{E}$. The proof of this lemma is routine and will therefore be omitted.

LEMMA 2. *Suppose:*

(i) $x_1, ..., x_n$ *are n distinct real numbers,*

(ii) $y_1, ..., y_{n+1}$ *are real numbers (not necessarily distinct),*

(iii) $L: \mathscr{P}_n \to \mathscr{R}$ *is a linear functional,*

(iv) $\omega(x) = (x - x_1) \cdots (x - x_n)$ *and* $L\omega \neq 0$.

Then there exists a unique polynomial, $q \in \mathscr{P}_n$, *such that* $q(x_k) = y_k$, $k = 1, ..., n$ *and* $Lq = y_{n+1}$.

Proof. If $q(x) = c_0 + c_1 x + \cdots + c_n x^n$ then $Lq = c_0(L1) + c_1(Lx) + \cdots + c_n(Lx^n)$, where Lx^k denotes the real number obtained by letting L act on the monomial, x^k. Therefore the coefficients, c_k, must satisfy the following $(n+1) \times (n+1)$ linear system of equations.

$$\begin{array}{ccccccccc} c_0 & + & c_1 x_1 & + & c_2 x_1^2 & + \cdots + & c_n x_1^n & = & y_1 \\ \vdots & & & & & & & & \vdots \\ c_0 & + & c_1 x_n & + & c_2 x_n^2 & + \cdots + & c_n x_n^n & = & y_n \end{array} \tag{2.1}$$
$$c_0(L1) + c_1(Lx) + c_2(Lx^2) + \cdots + c_n(Lx^n) = y_{n+1}.$$

In order to show that this system is solvable we first consider the function, f, defined by

$$f(x) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}.$$

Expanding this determinant we obtain that $f(x) = A_0 + A_1 x + \cdots + A_n x^n$, where A_k is the cofactor of the entry, x^k, in the last row. In particular, A_n is the Vandermonde detrminant for the points $x_1, ..., x_n$, and so $A_n \neq 0$. Since $f \in \mathscr{P}_n$ and f has zeros at $x_1, ..., x_n$, we obtain that $f(x) = A_n \omega(x)$. It follows that $A_n(L\omega) = Lf = A_0(L1) + A_1(Lx) + \cdots + A_n(Lx^n)$. This last sum is the expansion by cofactors of the last row for the determinant of the coefficient matrix in (2.1). Therefore (2.1) is uniquely solvable since $L\omega \neq 0$.

LEMMA 3. *Let $w \in \mathcal{W}$ and suppose $p \in \mathcal{P}_n^*$ is extremal for μ_n, $n \geqslant 1$. If wp has exactly n e-points, $x_1 < \cdots < x_n$, and $\omega(x) = (x - x_1) \cdots (x - x_n)$ then $(w\omega)'(t_0) = 0$ whenever t_0 is an e-point of $(wp)'$.*

Proof. Suppose to the contrary that $(wp)'$ has an e-point, t_0, such that $(w\omega)'(t_0) \neq 0$. Applying Lemma 2 with $Lf := (wf)'(t_0)$, we obtain a polynomial, $q \in \mathcal{P}_n$, such that $q(x_k) = \operatorname{sgn}[p(x_k)]$, $1 \leqslant k \leqslant n$, and $(wq)'(t_0) = -\operatorname{sgn}[(wp)'(t_0)]$. For sufficiently small $\varepsilon > 0$, $\|w(p - \varepsilon q)\| < \|wp\|$ (see remark after Lemma 1). Furthermore, $\|[w(p - \varepsilon q)]'\| \geqslant |(wp)'(t_0) - \varepsilon(wq)'(t_0)| > |(wp)'(t_0)| = \|(wp)'\|$. Therefore the ratio, $\|(wp)'\|/\|wp\|$, would become larger if p were replaced by $c(p - \varepsilon q)$ for any $c \in \mathcal{R}$, $c \neq 0$. This would contradict the fact that p is extremal for μ_n.

LEMMA 4. *Let $w \in \mathcal{W}$ and suppose $p \in \mathcal{P}_n^*$ is extremal for λ_n, $n \geqslant 1$. If wp has exactly n e-points, $x_1 < \cdots < x_n$, and $\omega(x) = (x - x_1) \cdots (x - x_n)$ then $\omega'(t_0) = 0$ whenever t_0 is an e-point of wp'.*

Proof. Suppose to the contrary that wp' has an e-point, t_0, such that $\omega'(t_0) \neq 0$. Applying Lemma 2 with $Lf := f'(t_0)$, and arguing in a manner similar to the proof of Lemma 3, we would obtain a polynomial, $q \in \mathcal{P}_n$, such that for sufficiently small $\varepsilon > 0$, $\|w(p - \varepsilon q)'\|/\|w(p - \varepsilon q)\| > \|wp'\|/\|wp\|$. This would contradict the fact that p is extremal for λ_n.

LEMMA 5. *Suppose $w \in \mathcal{W}$ and w'/w is decreasing on $\mathcal{R}$. Also suppose $p \in \mathcal{P}_n$ has n distinct real zeros. Then there are exactly $n+1$ distinct real numbers, where $(wp)'$ vanishes, and so wp can have at most $n+1$ e-points. Moreover if wp does have $n+1$ e-points then these e-points form an alternant for wp (i.e., $p = cT_n$, $c \neq 0$).*

Proof. Write $p(x) = c(x - z_1) \cdots (x - z_n)$, where $z_1 < \cdots < z_n$. The zero set of $(wp)'$ is the solution set of the equation,

$$\frac{w'(x)}{w(x)} = \sum_{k=1}^{n} \frac{1}{z_k - x}. \tag{2.2}$$

The argument that follows is easily motivated by considering the graph of the function, h, where $h(x)$ is the right side of (2.2). Note that h is continuous and increases from $-\infty$ to ∞ on each interval, (z_k, z_{k+1}), $k = 1, ..., n-1$. So (2.2) has one solution in each of these intervals. Also note that h is continuous on the interval, $(-\infty, z_1)$. Furthermore h maps this interval onto $(0, \infty)$. Therefore (2.2) must have a single solution in $(-\infty, z_1)$ unless $w'(x) < 0$ for all $x \in (-\infty, z_1)$. This last possibility is ruled out since $w(x) \to 0$ as $x \to -\infty$. A similar argument shows that (2.2) also has a single solution in the interval, (z_n, ∞). Thus we have shown that the solution set of (2.2) contains exactly $n+1$ points. Now suppose that each of these solutions, $x_1 < \cdots < x_{n+1}$, is an e-point of wp. If these e-points did not form an alternant for wp then $(wp)(x_{k+1}) = (wp)(x_k)$ for some k, $1 \leqslant k \leqslant n$. Therefore $(wp)'$ would have a zero in (x_k, x_{k+1}). However, since $x_1, ..., x_{n+1}$ are zeros of $(wp)'$, this would imply that $(wp)'$ vanished at more than $n+1$ points.

3. Proofs of Theorems

Proof of Theorem 1. Let $w \in \mathscr{W}$ and suppose $p \in \mathscr{P}_n^*$ is extremal for μ_n, $n \geqslant 2$. Let $a = a_n$ and $b = b_n$ so that for all $q \in \mathscr{P}_n$, $\|wq\| = \max\{|w(x)\, q(x)|: a \leqslant x \leqslant b\}$. Let t_0 be any e-point of $(wp)'$ and let $h(x) = (x - t_0)^2$. Note that t_0 cannot be an e-point of wp and hence $h(x) > 0$ whenever x is an e-point of wp. We first show that wp must have both $(+)$ points and $(-)$ points. (An e-point, x_0, of a function, f, is designated a $(+)$ point or a $(-)$ point according as $f(x_0) = \|f\|$ or $f(x_0) = -\|f\|$.) To see this suppose that wp had only $(+)$ points. Then for sufficiently small $\varepsilon > 0$, $\|w(p - \varepsilon h)\| < \|wp\|$. Moreover, $\|[w(p - \varepsilon h)]'\| \geqslant |(wp)'(t_0) - \varepsilon(wh)'(t_0)| = |(wp)'(t_0)| = \|(wp)'\|$. Therefore the ratio, $\|(wp)'\|/\|wp\|$, would become larger if p were replaced by $c(p - \varepsilon h)$ for any $c \in \mathscr{R}$, $c \neq 0$. However, this would contradict the fact that p is extremal for μ_n. Therefore wp must have both $(+)$ points and $(-)$ points. We assume that the smallest e-point of wp is a $(+)$ point. (If this were not the case the following argument would be modified in an obvious way.) By following the standard proof of the Tschebyscheff Equioscillation Theorem (cf. [2] or [5]) we can choose a finite number of points, $t_1 < \cdots < t_m$, in (a, b), none of which are e-points of wp, so that:

$[a, t_1]$ contains e-points of wp all of which are $(+)$ points,

$[t_1, t_2]$ contains e-points of wp all of which are $(-)$ points,

$\vdots$

$[t_m, b]$ contains e-points of wp all of which are $(+)$ points or $(-)$ points according as m is even or odd.

Since a maximal alternant for wp is clearly of size $m+1$, we need to show that $m+1 \geqslant n$. Let $g(x) = (t_1 - x) \cdots (t_m - x)\, h(x)$. Observe that $p(x)\, g(x) > 0$ whenever x is an e-point of wp. Hence for sufficiently small $\varepsilon > 0$, $\|w(p - \varepsilon g)\| < \|wp\|$. Moreover, $\|[w(p - \varepsilon g)]'\| \geqslant |(wp)'(t_0) - \varepsilon(wg)'(t_0)| = |(wp)'(t_0)| = \|(wp)'\|$. As before this would contradict the extremal nature of p unless $g \notin \mathscr{P}_n$. Therefore $\deg(g) = m + 2 \geqslant n + 1$. This establishes (i). To prove (ii) we first note that, because of (i), there are only three cases to consider: (1) $\deg(p) = n - 1$ and a maximal alternant for wp is of size n (i.e., $p = T_{n-1}$), (2) $\deg(p) = n$ and a maximal alternant for wp is of size $n+1$ (i.e., $p=T_n$), (3) $\deg(p)=n$ and a maximal alternant for wp is of size n. In the first two cases it follows immediately from Lemma 5 that a maximal alternant for wp consists of all e-points of wp, and hence is unique. In the third case p has at least $n-1$ distinct real zeros, $z_1, ..., z_{n-1}$. Since p is real there is one more real zero, z_n. Moreover if z_n were not distinct from $z_1, ..., z_{n-1}$ then wp would change sign at only $n-2$ places and so a maximal alternant would be of size $\leqslant n-1$. Therefore p has n distinct real zeros. Again Lemma 5 implies that the n points in a maximal alternant for wp are the only e-points of wp, and hence this maximal alternant is unique. The remainder of (ii) follows immediately from Lemma 3.

Proof of Theorem 2. Let $w \in \mathscr{W}$ and suppose $p \in \mathscr{P}_n^*$ is extremal for λ_n, $n \geqslant 1$. Suppose wp had only (+) points. Then it is easy to see that the ratio, $\|wp'\|/\|wp\|$, would become larger if p were replaced by $p-\varepsilon$ for some sufficiently small $\varepsilon>0$. Therefore wp must have both (+) points and (−) points. When $n=1$ this implies that $p=T_1$. We assume hereon that $n \geqslant 2$. Let $a, b, t_1, ..., t_m, g$, and h be as described in the proof of Theorem 1, except that in the definition of $h(x)$ we choose t_0 to be an e-point of wp'. We also choose the t_i's so that $t_0 \notin \{t_1, ..., t_m\}$. Clearly $p(x)\, g(x) \geqslant 0$ whenever x is an e-point of wp (strict inequality might not hold since t_0 could be an e-point of wp). It is also easy to see that $p(x)\, g(x) \geqslant 0$ when x is in a sufficiently small neighborhood of any e-point of wp. Therefore, by Lemma 1, there exists $\varepsilon>0$ so that $\|w(p-\varepsilon g)\| \leqslant \|wp\|$. Furthermore, $\|w(p-\varepsilon g)'\| \geqslant |w(t_0)[p'(t_0)-\varepsilon g'(t_0)]| = |w(t_0)\, p'(t_0)| = \|wp'\|$. The inequality in this chain can be made strict if t_0 is not an e-point of $w(p-\varepsilon g)'$. That this is indeed the case is easily seen by noting that the derivative of $w(p-\varepsilon g)'$, evaluated at t_0, is equal to $-\varepsilon w(t_0)\, g''(t_0) \neq 0$. Again the extremal nature of p requires that $\deg(g)=m+2 \geqslant n+1$. This establishes (i). The proof of (ii) can be obtained as in the proof of Theorem 1, except that Lemma 4 is used instead of Lemma 3.

Proof of Theorem 3. Let $w(x)=\exp(-x^2)$ and suppose $p \in \mathscr{P}_n^*$ is extremal for μ_n, $n \geqslant 1$. First note that either $\deg(p)=n-1$ or $\deg(p)=n$, and if $\deg(p)=n-1$ then $p=T_{n-1}$. If $n=1$ these statements are trivial and if $n \geqslant 2$ they follow from (i) of Theorem 1. From hereon we assume that $n \geqslant 1$ and $\deg(p)=n$. It remains to be shown that under these conditions, $p=T_n$. We begin by noting that p has n distinct real zeros. For $n=1$ this is clear and for $n \geqslant 2$ it was established in the proof of Theorem 1, part (ii). By Lemma 5, $(wp)'$ vanishes at exactly $n+1$ points, $x_1 < \cdots < x_{n+1}$, and by (i) of Theorem 1, at least n of these are e-points of wp. We now claim that all the points, $x_1, ..., x_{n+1}$, are e-points of wp. To see this suppose, to the contrary, that wp had only n e-points. Let ω be the monic polynomial of degree n whose zeros are at the e-points of wp, and let x_m, $1 \leqslant m \leqslant n+1$, denote that zero of $(wp)'$ which is not an e-point of wp. Since $(wp)'(x) = w(x)[p'(x) - 2xp(x)]$, it follows that $2xp(x) - p'(x) = 2(x - x_1) \cdots (x - x_{n+1}) = 2(x - x_m)\,\omega(x)$. Therefore $(w\omega)(x) = (wp)'(x)/2(x_m - x)$ and

$$(w\omega)'(x) = \frac{(x_m - x)(wp)''(x) + (wp)'(x)}{2(x_m - x)^2}.$$

Clearly if t_0 is an e-point of $(wp)'$ then $(w\omega)'(t_0) = (wp)'(t_0)/2(x_m - t_0)^2 \neq 0$, a result which contradicts Lemma 3. Therefore all of the points, $x_1, ..., x_{n+1}$, must be e-points of wp. By Lemma 5, these e-points are an alternate for wp, from which it follows that $p = T_n$.

4. Remarks

It seems likely that the conclusion of Theorem 3 could be improved by showing that T_{n-1} cannot be extremal for μ_n. This would be equivalent to showing that $\mu_1 < \mu_2 < \cdots$, or, more directly, by showing that $\|(wT_{n-1})'\|/\|wT_{n-1}\| < \|(wT_n)'\|/\|wT_n\|$, $n \geqslant 1$. This latter inequality can be confirmed by direct computation for $n = 1, 2$. For this purpose we note that $T_0(x) = 1$, $T_1(x) = x$, and $T_2(x) = x^2 - a$ where a is that number such that $a[\exp(a+1)] = 1$.

It would also be of interest to find other weights for which T_n is the extremal polynomial for either μ_n or λ_n.

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